Clearly, dV/dt will be < 0 along any perturbed motion if Q is positive definite and P is positive definite for all $t > t_0$ and all $x\epsilon[0, 1]$, or if the following inequalities are satisfied:

$$c_0 < 4\left[1 + \frac{m(1)}{\rho_a ab}\right]^{-1} \tag{13}$$

$$\left.\begin{aligned} &2(1-\alpha)\left(\min_x\left|\frac{dEI(x)}{dx}\right|\right)\left[lk_d(t,x) + \frac{1}{2}c_0v_0\frac{dm(x)}{dx}\right] > c_0v_0l^4k_d{}^2(t,x) \\ &(1-\alpha)\left[2c_0\alpha\left(lk_d(t,x) + \frac{1}{2}v_0\frac{dm(x)}{dx}\right)\left(\min_x\left|\frac{dEI(x)}{dx}\right|\right) - v_0{}^3(\pi\rho_a lab)^2\right] - \alpha v_0c_0{}^2l^4k_d{}^2(t,x) > 0 \end{aligned}\right\} \begin{array}{l}\text{for all } t > t_0 \\ \text{all } x\epsilon[0, 1]\end{array} \tag{14}$$

The preceding inequalities are deduced from those of Sylvester for positive definiteness of symmetric matrices. By choosing a constant c_0 satisfying both (7) and (13), the inequalities (11) and (14) become a sufficient condition for asymptotic stability. Note that since $(1 - \alpha)v_0{}^3(\pi\rho_a lab)^2 > 0$, only the second inequality in (14) needs to be considered. For the special case where the distributed damping coefficient $k_d \equiv 0$, α and c_0 may be taken to be

$$\alpha = \tfrac{1}{2} \qquad c_0 = (4 - \epsilon)\{1 + [m(1)/\rho_a ab]\}^{-1}$$

where ϵ is any positive number <4. Thus, condition (14) reduces to

$$v_0 < (2\pi l)^{-1}\left[(4-\epsilon)\rho_a{}^{-1}\frac{dm(x)}{dx}\left(\min_x\left|\frac{dEI(x)}{dx}\right|\right)\right]^{1/2} \times [ab(\rho_a ab + m(1))]^{-1/2} \tag{15}$$

The region of asymptotic stability in the v_0 - (ab) parameter plane represented by (15) with a typical set of parameters is shown in Fig. 2. The result agrees with physical reasoning that the stability boundary for v_0 should tend to $+\infty$ as the tail surface area $(ab) \rightarrow 0$. Since the exact stability boundary for this system is not known,‡ nothing can be said about the sharpness of condition (15).

Finally, it should be remarked that the selection of the functional V was by no means a straightforward task. In fact, among the class of quadratic functionals of the state variables, only (5) was found to be useful. Unfortunately, there are no systematic methods for constructing the required functionals starting from the system equations. This, of course, is the main difficulty in applying Lyapunov's direct method. On the other hand, sufficient conditions for asymptotic stability of equilibrium of more complex flexible vehicles have been obtained using the approach outlined here.[4] Since this method is valid for a rather wide class of dynamical systems, further investigations on its applications to aeroelastic stability problems may be fruitful.

References

[1] Wang, P. K. C., "Application of Lyapunov's direct method to stability problems in elastic and aeroelastic systems," IBM Research Rept. RJ-305 (June 1964).

[2] Hahn, W., *Theory and Application of Liapunov's Direct Method* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1963).

[3] Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., *Aeroelasticity* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955).

[4] Wang, P. K. C., "Stability analysis of a simplified flexible aerodynamic vehicle with pitch autopilot via Lyapunov's direct method," IBM Research Lab., San Jose, Calif., Final Tech. Documentary Report, U. S. Air Force Contract AF 33(657)-11545 (1965).

‡ Note that a stability analysis based on an approximate finite-dimensional mathematical model of (1–3) generally leads to conditions that are insufficient for asymptotic stability.

A Single Formula for the Velocity Distribution in the Turbulent Inner and Outer Boundary Layers

HWACHII LIEN*
Avco Corporation, Wilmington, Mass.

Introduction

In applying the integral method for determining turbulent boundary-layer gross properties, it is necessary to know the velocity distribution in the viscous sublayer and turbulent inner and outer layers. More than a dozen investigators have proposed various analytical forms for the velocity profile, but none of them offer a simple single formula that can be used directly in the integral method. The nature of these various profiles is reviewed and summarized in the survey articles by Kestin and Richardson,[1] and Spalding and Chi.[2] It must be noted that all the investigators, except Reichardt,[3] van Driest,[4] and Spalding,[5] introduced two or more separate expressions in order to describe the velocity distribution in the entire three regions.

For the convenience of integration across the boundary layer, it is desirable to express the velocity profile by one simple mathematical formula rather than by two or more discontinuous formulas. However, both van Driest's and Spalding's formulas are impractical for the direct application of the integral method because the former involves the derivative dy_*/du_*, and the latter gives y_* as a function of u_* instead of vice versa. Moreover, Reichardt's and Spalding's velocity profiles, as shown in Figs. 1 and 2, do not satisfy the defect law in the outer portion of the turbulent boundary layer. Since the major portion of the momentum in the boundary layer is contained in this turbulent outer layer, Reichardt's and Spalding's formulas are not satisfactory for the solution of momentum integral equation.

A more suitable velocity profile in the form that can be used in the integral method is recently derived by Libby et al.[6] Their result does describe both "the law of the wall" and "the defect law" adequately, but, three different forms including a complicated expression for the outer layer are required to represent the velocity profile completely.

Instead of using an analytical expression, Coles[7] has succeeded in describing the complete velocity profile with a numerically tabulated wake function. To achieve the analytical integration of the momentum equation across the boundary-layer thickness, some modifications of Coles' approach are attempted in the present study.

Formulation and Derivation

Based upon the mathematical and physical arguments, the new expression should be smooth and continuous over the region of the boundary layer with an analytical form permitting easy integration across the boundary-layer thickness. It should satisfy the conditions that 1) the normalized

Received April 5, 1965; revision received June 21, 1965.

* Senior Staff Scientist, Aerophysics Department. Member AIAA.

velocity profile u/u_τ, where $u_\tau = (\tau_w/\rho)^{1/2}$, is equal to $u_\tau y/\nu$ in the viscous sublayer; 2) the profile u/u_τ is a function of $u_\tau y/\nu$ alone in the turbulent inner layer (the law of the wall); and 3) the profile u/u_τ is a function of both $u_\tau y/\nu$ and y/δ in the turbulent outer layer (the defect law).

With the entire turbulent boundary layer being divided into three subregions, there are at least six boundary conditions (two at the wall, two at the edge of the boundary layer, and one edge condition each at the two interfaces of subregions) that must be satisfied by the formula for the new velocity profile. This requirement made the simplification of the problem nearly impossible, and it is considered more advantageous to treat the turbulent region and the viscous subregion separately. To this end, the velocity profile is assumed to take the forms

$$u/u_e = \tilde{\tau}_w\eta \qquad \tilde{\tau}_w \equiv \tau_w\delta/\mu u_e \qquad \eta \equiv y/\delta \tag{1}$$

in the viscous subregion and

$$u/u_e \equiv f = c\ln\eta + a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 \tag{2}$$

in the entire turbulent region. It must be noted that Eq. (1) is derived from the physical argument that the viscosity and shear stress are constant in the viscous sublayer, and Eq. (2) is the linear superposition of the defect law and the law of the wall. Denoting $u_* \equiv u/u_\tau$, $y_* \equiv u_\tau y/\nu$, and $R_\tau \equiv u_\tau\delta/\nu$, the five coefficients that appeared in Eq. (2) can be determined from the conditions that

$$u_* = A\ln y_* + B \qquad \text{at } \eta_s \leq \eta \leq \eta_1 \tag{3}$$

$$\partial f/\partial\eta = \tau_1\delta/\epsilon u_e \equiv \tilde{\tau}_1 = R_\tau(u_\tau/u_e)(\tau_1/\tau_w)(\mu/\epsilon) \qquad \text{at } \eta = \eta_1 \tag{4}$$

and

$$f = 1 \qquad \partial f/\partial\eta = 0 \qquad \text{at } \eta = 1 \tag{5}$$

where A and B in Eq. (3) are known constants.[7] The subscripts 1, e, s, and w represent, respectively, the conditions at the interface of the inner and the outer layers, the edge of the outer layer, the edge of the viscous sublayer, and the wall. Combining these boundary conditions with Eq. (2) yields five algebraic equations from which the five coefficients can be determined. In solving these five equations, it is necessary to know the shear stress ratio τ_1/τ_w and the ratio of laminar and eddy viscosities μ/ϵ. An equation relating these two quantities can be derived from the expression for the shear stresses at the interface of turbulent inner and outer layers. Since

$$\frac{\tau_1}{\tau_w} = \frac{\epsilon\cdot\partial u/\partial y|_{\eta=\eta_1}}{\tau_w} = \frac{\epsilon u_e}{\delta\rho u_\tau^2}\cdot\frac{\partial f}{\partial\eta}\bigg|_{\eta=\eta_1} = \frac{\mu}{\delta u_\tau\rho}\,\frac{\epsilon}{\mu}\,\frac{A}{\eta_1}$$

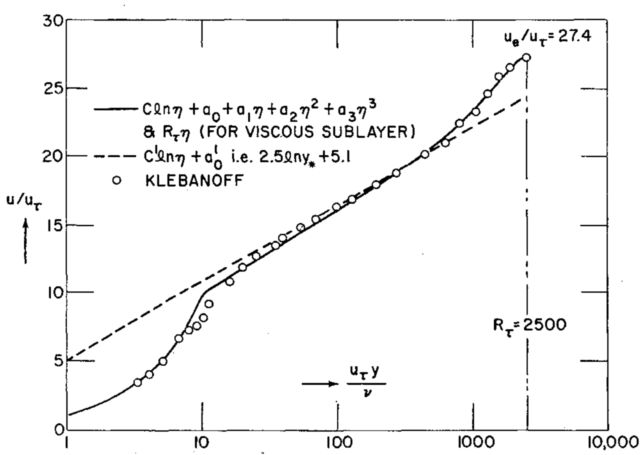

Fig. 1 Discrepancy between derived profiles and experimental data.

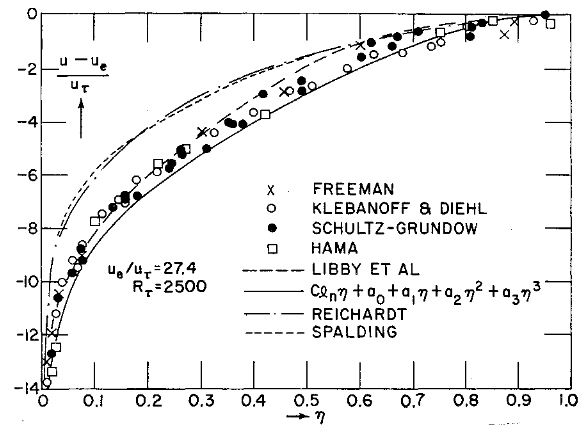

Fig. 2 Comparison of velocity profiles.

therefore,

$$(\tau_1/\tau_w)(\mu/\epsilon) = 1/R_\tau\cdot A/\eta_1 \tag{6}$$

Neglecting the terms that contain η_s^2 and higher orders, the solutions become

$$C = A/(u_e/u\tau), \qquad \left.\begin{aligned} a_0 &= (u_s/u_e) - C\ln\eta_s - a_1\eta_s \\ a_1 &= -\eta_1/s[C(2-3\eta_1) + 6Q(1-\eta_1)] \\ a_2 &= 1/s[(1-3\eta_1^2)C + 3(1-\eta_1^2)Q] \\ a_3 &= -1/s[C(1-2\eta_1) + 2Q(1-\eta_1)] \end{aligned}\right\} \tag{7}$$

where

$$Q = 1 - (u_s/u_e) + C\ln\eta_s \qquad \text{and} \qquad S = 1 - \eta_1(4 - 3\eta_1)$$

Thus, the velocity profile in the turbulent region is completely specified by Eqs. (1) and (2) together with (7).

Example and Discussion

Based upon the results of Refs. 6 and 7 for the constant pressure case, the universal limits

$$A = 2.5 \qquad u_s/u_\tau = 10.2 \qquad \eta_s = 11.5/R_\tau \qquad \eta_1 = 0.131 \tag{8}$$

can be considered as reasonable values for the application of the integral technique. Under this condition, the velocity profile $u/u_e(\eta)$ is uniquely determined from the given flow conditions R_τ and u_e/u_τ. According to Eqs. (8), the coefficients in the velocity profile

$$u/u_e = c\ln\eta + a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3$$

can be specified by

$$C = 2.5/(u_e/u\tau) \qquad \left.\begin{aligned} a_0 &= 1 - Q - (11.5/R_\tau)a_1 \\ a_1 &= -0.405(C + 3.25Q) \\ a_2 &= 1.805(C + 3.11Q) \\ a_3 &= -1.405(C + 2.36Q) \end{aligned}\right\} \tag{9}$$

with

$$Q = 1 - 10.2/(u_e/u_\tau) + C\ln(11.5/R_\tau)$$

The velocity profile for $u_e/u_\tau = 27.4$ and $R_\tau = 2500$, as derived from the present approach, is compared with the corresponding velocity profiles obtained by other investigators. Although the theoretical results of the present study and of Libby et al. are shown, in Fig. 2, to be in fair agreement with the experimental data, both Spaulding's and Reichardt's are again demonstrated to be inaccurate at the outer layer.

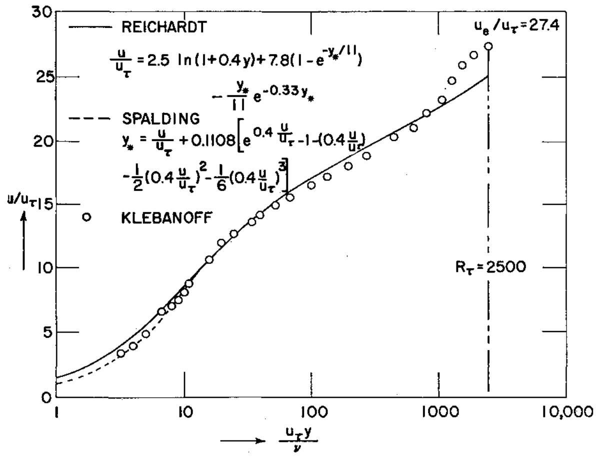

Fig. 3 Velocity profiles at R_T = 2500.

A similar comparison, plotted in different coordinates, with the experimental data of Klebanoff as presented by Coles in Ref. 7 is shown in Fig. 3. Although it is not shown here, the velocity profiles derived from the present theory are also compared with the experimental profiles at different R_τ and u_e/u_τ and are proved to be quite satisfactory at a wide range of flow conditions.

It must be noted that the turbulent boundary-layer problem can, in the present state of knowledge, be attacked only by semiempirical method. The extent to which the analysis is simplified depends on how far it appears possible to derive or interpret, in an empirical manner, general relations for the unknown quantities from available experimental data. Therefore, it is not surprising that the various investigators have quite different viewpoints with regards to the simplification of the problem. In this connection, one may attempt to simplify the present approach somewhat further by neglecting the small portion of contribution to the momentum integral from the velocity profile of Eq. (1). That is, Eq. (2) may be used throughout the entire region of the boundary layer as long as the integral approximation is applied for the solution of the gross quantities (such as the momentum thickness). The boundary conditions at the wall, however, must be evaluated from Eq. (1). In conclusion, it may be stated that the single formula for the velocity profile derived in the present analysis provides a relatively simple, consistent, and fairly accurate means of computing the quantitative descriptions of the turbulent boundary layer.

References

[1] Kestin, J. and Richardson, P. D., "Heat transfer across turbulent incompressible boundary layers," Intern. J. Heat Mass Transfer **6**, 147–189 (1963).

[2] Spalding, D. B. and Chi, S. W., "The drag of a compressible turbulent boundary layer on a smooth flat plate with and without heat transfer," J. Fluid Mech. **9**, 117–143 (1964).

[3] Reichardt, H., "Die Grundlagen des Turbulenten Warneuberganges," Arch. Ges. Warmetech. **2**, 129–142 (1951).

[4] van Driest, E. R., "On turbulent flow near a wall," J. Aeronaut. Sci. **23**, 1007–1011 (1956).

[5] Spalding, D. B., "A single formula for the 'law of the wall,'" J. Appl. Mech. **28**, 455–457 (1961).

[6] Libby, P. A., Baronti, P. O., and Napolitano, L., "Study of the incompressible turbulent boundary layer with pressure gradient," AIAA J. **2**, 445–452 (1964).

[7] Coles, D., "The law of the wake in the turbulent boundary layer," J. Fluid Mech. **1**, 191–226 (1956).

Stepped Circular Kirchhoff Plate

WALTER D. PILKEY*

IIT Research Institute, Chicago, Ill.

THE method of initial parameters is becoming increasingly popular as an approach for solving the deformation problems of classical advanced strength of materials.[1] Relative to many familiar solution schemes, the century-old method of initial parameters involves algebraically complicated expressions, a fact that dampened enthusiasm for the approach until the widespread acceptance of the digital computer as an engineering tool. The method is characterized by the favorable properties of broad scope in covering numerous problems, ease in applicability to problems with complicated loadings or boundary conditions, and appeal to engineers wishing to avoid complex mathematical manipulations and preferring to retain visible control of the physics of the problem.

The present work is intended to contribute the axially symmetric stepped circular plate to the growing catalog of initial parameter solutions. Solutions have already been established for most conceivable forms of the Euler-Bernoulli beams[2] and for some circular plates, rectangular plates, cylinders, rotationally symmetric shells, and arches. In addition, these fundamental solutions for structural members have been extended in bona fide initial parameter form to the multiple statically indeterminate case of an unlimited number of intermediate conditions, including discrete generalized elastic springs or rigid supports (that is, continuous beams in the case of beams). Much of this work reached a well-developed state more than a score of years ago in the Soviet Union.[3] Today, solutions in this field appear in German and British literature under such titles as "transfer matrices" and extensions of "Macauley's Method," respectively.

The axially symmetric flexure of a circular plate of stepped cross section can be handled with classical methods[4,5] by matching appropriate physical variables at each side of each step of a sectionally applied solution (such as the initial parameter solution[6]) for the plate of constant cross section. However, the current work yields for a plate with unlimited steps a true initial parameter solution that can easily be programmed for a digital computer. Advantage can be taken of the usual benefits of initial parameter solutions such as the fully tabulated values (in terms of the initial parameter functions and loading functions that vary from member to member) of the initial parameters themselves.

The fundamental equations describing axially symmetric bending motion of a circular plate that is subject to the assumptions of Kirchhoff's plate theory are given by[7] 1) the loading intensity in force per unit area

$$q(r) = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{D}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} - \frac{1}{r}\frac{d}{dr}\left[r(1-\mu)\frac{dD}{dr}\frac{1}{r}\frac{dw}{dr}\right] \quad (1)$$

2) the total shearing force on a cylindrical section of radius r

$$V(r) = 2\pi r\left\{\frac{d}{dr}\left[D\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] - (1-\mu)\frac{dD}{dr}\left(\frac{1}{r}\frac{dw}{dr}\right)\right\}$$

3) the total radial moment on a cylindrical section of radius r

$$M(r) = -2\pi r D\left[(d^2w/dr^2) + (\mu/r)(dw/dr)\right]$$

Received April 12, 1965; revision received May 5, 1965. The author acknowledges the assistance of Theodore Liber, Research Engineer, IIT Research Institute, in solving this problem.

* Research Engineer, Solid Mechanics Research.